Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of
MATHEMATICAL

www.elsevier.com/locate/jmaa

Bifurcating periodic solutions of wind-driven circulation equations

Zhi-Min Chen [a,b,*], W.G. Price [a]

[a] *School of Engineering Sciences, Ship Science, University of Southampton, Southampton SO17 1BJ, UK*
[b] *School of Mathematics, Nankai University, Tianjin 300071, People's Republic of China*

Received 5 June 2004

Available online 28 January 2005

Submitted by C.E. Wayne

Abstract

The existence of bifurcating periodic flows in a quasi-geostrophic mathematical model of wind-driven circulation is investigated. In the model, the Ekman number r and Reynolds number R control the stability of the motion of the fluid. Through rigorous analysis it is proved that when the basic steady-state solution is independent of the Ekman number, then a spectral simplicity condition is sufficient to ensure the existence of periodic solutions branching off the basic steady-state solution as the Ekman number varies across its critical value for constant Reynolds number. When the basic solution is a function of Ekman number, an additional condition is required to ensure periodic solutions.

Keywords: Hopf bifurcation; Periodic solution; Navier–Stokes equation; Bessel-potential space

1. Introduction

Bifurcation analysis plays a crucial role in understanding qualitative changes of flow regimes of oceanic and atmospheric circulation equations. From the view point of numer-

* Corresponding author.
E-mail address: zhimin@soton.ac.uk (Z.-M. Chen).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.09.062

ical experiments, bifurcation phenomena relating to ocean circulation were examined by Charney and DeVore [2], Veronis [19,20], and Pedlosky [15] to determine the occurrence of multi steady-state solutions, whereas Jiang et al. [9], Jin and Ghil [10] predicted periodic circulatory motions.

Because wind-driven circulation problems embody extremely complex physical mechanisms, to aid understanding it is beneficial to study simplified models (see Ghil and Childress [6], Lions et al. [13,14], and Pedlosky [16]). For example, Pedlosky [16] developed a suitable mathematical model describing mid-latitude wind-driven circulation through a quasi-geostrophic approximation. This model describes the dynamics of circulation flow-driven by a wind stress and influenced by Ekman friction layers, bottom topography, and a β-plane approximation of the Coriolis force.

The purpose of this study is to derive a general Hopf bifurcation theorem of the two-dimensional simplified wind-driven circulation equation (see Pedlosky [16, Eq. (5.2.22)]) which can be expressed in the following dimensionless vorticity formulation:

$$\partial_t \Delta\psi + r\Delta\psi - \frac{1}{R}\Delta^2\psi + \beta\partial_x\psi + J(\psi, \Delta\psi + \eta_B) = \beta \operatorname{curl}\tau \quad \text{in } \Omega \tag{1}$$

together with the free slip boundary condition

$$\psi = 0, \quad \Delta\psi = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Here $J(\psi, \phi) = \partial_x\psi\partial_y\phi - \partial_y\psi\partial_x\phi$ is the advection operator, $\psi = \psi(x, y, t)$ describes a geostrophic stream function, $\tau = (\tau_1(x, y), \tau_2(x, y))$ is a steady-state wind stress applied on the circulation basin Ω. The x-axis points eastwards and the y-axis northwards. The zonal and meridional velocity components u and v are given by

$$u = -\partial_y\psi, \qquad v = \partial_x\psi,$$

and the relative vorticity ζ is defined by $\zeta = \Delta\psi$.

The mathematical formulation (1) is a quasi-geostrophic approximation of the shallow water equation under the effect of a Coriolis force (see [16]). The parameter $\eta_B = \eta_B(x, y)$ is a function measuring the topography height of the bottom of the original fluid domain. Control parameters defining the fluid motion are Reynolds number R, the Ekman number r measuring the effects of friction arising from the top and bottom Ekman layers of the fluid, and the number β is derived from a β-plane approximation of the Coriolis force. The interested reader may find further descriptions of this equation and the theoretical background by consulting Pedlosky [16, pp. 260–261].

The existence of bifurcating periodic solutions branching off a basic steady-state solution of an ordinary differential equation system was first derived by Hopf [7,8]. The Hopf bifurcation theorem of Navier–Stokes equations is well-established under the assumption that the nonreal eigenvalue of the linearized spectral problem satisfies the simplicity condition and a condition relating to the transversal crossing of the imaginary axis at a critical value of Reynolds number (see Yudovich [22], Joseph and Sattinger [11]). Equation (1) is similar in form to the vorticity formulation of Navier–Stokes equations. Thus the Hopf bifurcation theorem derived in [11] is applicable to (1) to ensure the existence of bifurcating periodic solutions as the Reynolds number varies across a critical value R_c for constant Ekman number. However (1) also depends on the Ekman number r and this introduces an additional parameter to influence Hopf bifurcations. In this paper we are interested in the existence of bifurcating periodic solutions of (1) as the parameter r varies across an Ekman critical value $r_{\rm cr}$ under the constraint that the number β and the Reynolds number $R > 0$ are held constant. The occurrence of the Ekman number r, in fact, helps to simplify the Hopf bifurcation theorem. Through a rigorous analysis it is proved that the spectral simplicity condition is sufficient to ensure the existence of bifurcating periodic solutions provided that the basic steady-state solution is independent of Ekman number r. In this special circumstance, the spectral transversal crossing condition becomes irrelevant.

Although the developed proof in this study is based on the implicit function theorem, which was also used by Joseph and Sattinger [11], the bifurcation theorem herein is derived by working directly in the Bessel-potential function valued space defined in Section 3, rather than in the L_2 space $L_2(0, 2\pi; L_2(\Omega))$ and the Hölder space $C^{(3+\alpha)/2,3+\alpha}([0, 2\pi] \times \bar{\Omega}))$ examined in [11]. In order to use the implicit function theorem on the periodic functions in the space $L^2(0, 2\pi; H^4(\Omega))$, it is essential to establish an a priori estimate of the linearized evolutionary equation of (1) in the space $L^2(0, 2\pi; H^4(\Omega))$. If a Fourier expansion is used, it is necessary to establish an a priori estimate in the Bessel-potential space $H^4(\Omega)$ for the linearized stationary equation of (1) as discussed in Section 3. Thus, in the rigorous analysis of this study, neither the Hölder estimates of evolutionary Stokes equations (see Ladyzhenskaya [12]) nor the analytic semigroup theory (see [11]) is used. A comparison with the classical Hopf bifurcation theorem derived by Joseph and Sattinger [11] employing the implicit function theorem on the periodic functions in both the L_2 space $L_2(0, 2\pi; L_2(\Omega))$ and the Hölder space $C^{(3+\alpha)/2,3+\alpha}([0, 2\pi] \times \Omega)$ shows that they adopted a Fredholm decomposition and a Fourier expansion to derive the L_2-estimates of the linearized evolutionary equation of (1). This estimate enables the establishment of the unique solvability of periodic solutions of the linearized evolutionary equation of (1) in the L_2 space. Moreover, they used the Hölder estimates of the evolutionary Stokes operator [12], analytic semigroup theory, and the unique solvability in the L_2 space to derive the uniqueness and existence of periodic solutions of the linearized evolutionary equation in the Hölder space $C^{(3+\alpha)/2,3+\alpha}([0, 2\pi] \times \Omega)$ due to the fact that L_2-estimate is insufficient to control the periodic solutions of (1). Thus the proof of the present investigation, which is self-contained, seems more elementary and transparent.

In this paper, the driven force τ is independent of time and thus gives the existence of a basic steady-state solution. If the driven force is a periodic function, Duan [4] proved the existence of a basic periodic solution of (1), (2). Moreover, Duan and Kleoden [5] derived the existence of almost periodic and quasi-periodic solutions of the quasi-geostrophic fluid model when the wind-driven force is almost periodic and quasi-periodic for constant Ekman and Reynolds numbers.

The main result of this investigation is stated in Section 2. That is, the spectral simplicity condition ensures the existence of bifurcating periodic solutions when the basic steady-state solution is independent of the Ekman number r. Section 3 describes the decomposition of the 2π-period function space contained in $L_2(0, 2\pi; H^4(\Omega))$ with respect to the linearized evolutionary equation of (1), (2). In the $L_2(\Omega)$ rather than in the Bessel-potential space $H^4(\Omega)$, this decomposition approach is elementary and is described in [11]. Section 4 derives the proof of the main result which is an immediate consequence of the decomposition result derived in Section 3 and the implicit function theorem. In Section 5, we consider the case of basic steady-state solution relating to Ekman parameter r. In this circumstance, imposition of the spectral transversal condition is necessary to ensure the existence of bifurcating periodic solutions.

2. Statement of the theorem

To investigate wind-driven circulation behaviour and to simplify the mathematical model, we confine the bounded current basin domain Ω to be either smooth or rectangular, and let η_B and wind stress τ be smooth functions on $\Omega \cup \partial\Omega$, the closure of Ω. The basic steady-state solution of (1), (2) is therefore a smooth function and is denoted by ψ_0.

The substitution of solution $\psi = \psi_0 + \psi'$ in (1) and the omission of the subscript prime gives

$$\partial_t \Delta\psi + L\psi + J(\psi, \Delta\psi) = 0, \tag{3}$$

where the linear operator

$$L\psi = r\Delta\psi - \frac{1}{R}\Delta^2\psi + \beta\partial_x\psi + J(\psi_0, \Delta\psi) + J(\psi, \Delta\psi_0 + \eta_B).$$

The conjugate operator L^* of L is defined as

$$(L\psi, \phi) = (\psi, L^*\phi),$$

where the scalar product

$$(\psi, \phi) = \int_\Omega \psi(x, y)\bar{\phi}(x, y)\,dx\,dy,$$

whenever $\phi, \psi \in L_2(\Omega)$. The proof of existence of a Hopf bifurcation is essentially based on the occurrence of the critical eigenvalue $i\omega_{\mathrm{cr}}$ of the critical spectral equation

$$-i\omega_{\mathrm{cr}}\Delta\phi + L_{\mathrm{cr}}\phi = 0 \quad (\omega_{\mathrm{cr}} > 0), \tag{4}$$

where

$$L_{\mathrm{cr}}\psi = r_{\mathrm{cr}}\Delta\psi - \frac{1}{R}\Delta^2\psi + \beta\partial_x\psi + J(\psi_0, \Delta\psi) + J(\psi, \Delta\psi_0 + \eta_B)$$

and r_{cr} denotes a critical Ekman number such that the neutral spectral problem (2), (4) admits an eigenfunction.

In the present study R and β are assumed constant and it is shown that with changing values of r around a critical Ekman number r_{cr}, periodic solutions branch off the steady-state solution ψ_0.

The main result reads as follows.

Theorem 2.1. *Let ψ_0 be a classical steady-state solution of* (1), (2) *independent of the parameter r, and let the constants $\beta \in (-\infty, \infty)$, $R > 0$, $r_{\mathrm{cr}} > 0$ such that the spectral problem defined by* (2), (4) *admits a solution $(i\omega_{\mathrm{cr}}, \phi_0)$ which is simple in the following sense*:

$$\textit{Dimension of } \{\phi; -im\omega_{\mathrm{cr}}\Delta\phi + L_{\mathrm{cr}}\phi = 0\} = \begin{cases} 1, & \textit{when } m = 1, \\ 0, & \textit{when } m = 0, 2, 3, 4, \ldots, \end{cases}$$

$$\left(\phi_0, \Delta\phi_0^*\right) \neq 0,$$

where ϕ_0^* *denotes the conjugate eigenfunction of* ϕ_0 *such that*

$$\left(i\omega_{\mathrm{cr}}\Delta + L_{\mathrm{cr}}^*\right)\phi_0^* = 0. \tag{5}$$

Then (1), (2) *admit a real classical periodic solution* ψ_ε *of* $2\pi/\omega(\varepsilon)$ *period in the following form*:

$$\psi_\varepsilon = \psi_0 + \varepsilon\left[e^{-i\omega(\varepsilon)t}\phi_0 + e^{i\omega(\varepsilon)t}\bar\phi_0\right] + O\left(\varepsilon^2\right),$$
$$\psi_\varepsilon\left(x, y, t + 2\pi/\omega(\varepsilon)\right) = \psi_\varepsilon(x, y, t),$$
$$\omega(\varepsilon) = \omega_{\mathrm{cr}} + O(\varepsilon), \qquad r(\varepsilon) = r_{\mathrm{cr}} + O(\varepsilon),$$

provided that $\varepsilon > 0$ *is sufficiently small.*

3. Estimates in Bessel-potential spaces

Let us introduce the complex Bessel-potential spaces

$$H^4 = H^4(\Omega) = \left\{\phi \in L_2;\ \phi|_{\partial\Omega} = \Delta\phi|_{\partial\Omega} = 0,\ \Delta^2\phi \in L_2\right\},$$

where the complex space

$$L_2 = L_2(\Omega) \quad \text{with the norm } \|\phi\| = \left(\int_\Omega |\phi|^2\,dx\,dy\right)^{1/2}.$$

It is obvious that the eigenfunctions $\phi_0, \phi_0^* \in H^4$. Since $(\phi_0, \Delta\phi_0^*) \neq 0$, we may suppose $(\phi_0, \Delta\phi_0^*) = 1$. Noting that

$$i\omega_{\mathrm{cr}}\left(\phi_0, \Delta\bar\phi_0^*\right) = \left(L_{\mathrm{cr}}\phi_0, \bar\phi_0^*\right) = \left(\phi_0, \overline{L_{\mathrm{cr}}^*\phi_0^*}\right) = \left(\phi_0, i\omega_{\mathrm{cr}}\Delta\bar\phi_0^*\right) = -i\omega_{\mathrm{cr}}\left(\phi_0, \Delta\bar\phi_0^*\right),$$

we thus have the orthogonal property

$$\left(\phi_0, \Delta\bar\phi_0^*\right) = \left(\bar\phi_0, \Delta\phi_0^*\right) = 0, \qquad \left(\phi_0, \Delta\phi_0^*\right) = \left(\bar\phi_0, \Delta\bar\phi_0^*\right) = 1. \tag{6}$$

Lemma 3.1. *Let the projection operator*

$$P_m\psi = \begin{cases} \psi, & \textit{when } m \neq \pm 1, \\ \psi - (\psi, \Delta\phi_0^*)\phi_0, & \textit{when } m = 1, \\ \psi - (\psi, \Delta\bar\phi_0^*)\bar\phi_0, & \textit{when } m = -1. \end{cases}$$

Then for every $f_m \in L_2$, *the equation*

$$\left(-im\omega_{\mathrm{cr}} + \Delta^{-1}L_{\mathrm{cr}}\right)\psi_m = \Delta^{-1}f_m \tag{7}$$

admits a unique solution ψ_m *in the following sense*:

$$P_m\psi_m = \left(-im\omega_{\mathrm{cr}} + \Delta^{-1}L_{\mathrm{cr}}\right)^{-1}P_m\Delta^{-1}f_m \in H^4$$

and

$$\|\Delta^2 P_m\psi_m\| \leqslant c\|f_m\|$$

with the constant c independent of the function f_m and any integer m.

Proof. Due to the definitions of P_m and ϕ_0^*, we have the community property:

$$P_m(-im\omega_{\rm cr} + \Delta^{-1}L_{\rm cr}) = (-im\omega_{\rm cr} + \Delta^{-1}L_{\rm cr})P_m.$$

Applying the operators P_m and $R\Delta^{-1}P_m$ to (7) respectively, we derive

$$(-im\omega_{\rm cr} + \Delta^{-1}L_{\rm cr})P_m\psi_m = P_m\Delta^{-1}f_m \tag{8}$$

and

$$-P_m\psi_m + M_mP_m\psi_m = R\Delta^{-1}P_m\Delta^{-1}f_m \tag{9}$$

with

$$\begin{aligned} M_m\psi &\equiv -im\omega_{\rm cr}R\Delta^{-1}\psi + R\Delta^{-2}L_{\rm cr}\psi + \psi \\ &= R(-im\omega_{\rm cr} + r_{\rm cr})\Delta^{-1}\psi + R\Delta^{-2}\big(\beta\partial_x\psi + J(\psi_0, \Delta\psi) + J(\psi, \Delta\psi_0 + \eta_B)\big). \end{aligned}$$

Since

$$\begin{aligned} &\|\Delta^2 M_m\psi\| \\ &\quad\leqslant R\big(|m|\omega_{\rm cr} + r_{\rm cr}\big)\|\Delta\psi\| + R\big\|\beta\partial_x\psi + J(\psi_0, \Delta\psi) + J(\psi, \Delta\psi_0 + \eta_B)\big\| \\ &\quad\leqslant c\big(|m|\|\Delta\psi\| + \|\nabla\psi\| + \|\nabla\Delta\psi\|\big), \end{aligned}$$

the operator $M_m : H^4 \mapsto H^4$ is compact. By the Fredholm alternative principle (see, for example, [21]), Eq. (9) has a unique solution $P_m\psi_m \in H^4$ if and only if

$$\big(\Delta^2\Delta^{-1}P_m\Delta^{-1}f_m, \Delta^2\hat\phi_m\big) = 0, \tag{10}$$

where $\hat\phi_m$ is the solution of the equation

$$-\hat\phi_m + \hat M_m\hat\phi_m = 0.$$

Here $\hat M_m$ denotes the conjugate operator of M_m in the scalar product of H^4 and defined in the sense:

$$\big(\Delta^2 M_m\psi, \Delta^2\phi\big) = \big(\Delta^2\psi, \Delta^2\hat M_m\phi\big).$$

This together with the definition of the operator M_m and an integration by parts yields

$$-\phi + \hat M_m\phi = R\Delta^{-2}\big(im\omega_{\rm cr}\Delta + L^*_{\rm cr}\big)\Delta^2\phi.$$

It follows from the simplicity condition that

$$\Delta^2\hat\phi_m = \begin{cases} 0, & \text{when } m \neq \pm 1, \\ \phi_0^*, & \text{when } m = 1, \\ \bar\phi_0^*, & \text{when } m = -1. \end{cases}$$

This implies the validity of (10) for any integer m and also the inverse of the operator $-1+M_m$. We thus have

$$P_m\psi_m = R(-1+M_m)^{-1}\Delta^{-1}P_m\Delta^{-1}f_m$$

and

$$\|\Delta^2 P_m\psi_m\| = R\|\Delta^2(-1+M_m)^{-1}\Delta^{-1}P_m\Delta^{-1}f_m\| \leqslant c_m\|\Delta P_m\Delta^{-1}f_m\| \leqslant c_m\|f_m\|.$$

Now it remains to prove that the constants c_m are bounded or the constants c_m with $m\neq\pm1$ are bounded. To do this, we apply Δ to (8) and take a scalar product of the resulting equation with $\Delta^2\psi_m$ to produce

$$\big(-im\omega_{\mathrm{cr}}\Delta\psi_m + L_{\mathrm{cr}}\psi_m, \Delta^2\psi_m\big) = \big(f_m, \Delta^2\psi_m\big) \quad \text{for } m\neq\pm1,$$

where we have used the result that $P_m\psi_m = \psi_m$ for $m\neq\pm1$ because of the definition of P_m. We thus have

$$\begin{aligned} &im\omega_{\mathrm{cr}}\|\nabla\Delta\psi_m\|^2 - r_{\mathrm{cr}}\|\nabla\Delta\psi_m\|^2 - \frac{1}{R}\|\Delta^2\psi_m\|^2 \\ &\quad = \big(f_m, \Delta^2\psi_m\big) - \big(\beta\partial_x\psi_m + J(\psi_0, \Delta\psi_m) + J(\psi_m, \Delta\psi_0+\eta_B), \Delta^2\psi_m\big) \end{aligned}$$

and so

$$\begin{aligned} &|m|\omega_{\mathrm{cr}}\|\nabla\Delta\psi_m\|^2 + r_{\mathrm{cr}}\|\nabla\Delta\psi_m\|^2 + \frac{1}{R}\|\Delta^2\psi_m\|^2 \\ &\quad \leqslant 2\|f_m\|\|\Delta^2\psi_m\| + 2\|\beta\partial_x\psi_m + J(\psi_0,\Delta\psi_m) + J(\psi_m, \Delta\psi_0+\eta_B)\|\|\Delta^2\psi_m\| \\ &\quad \leqslant 2\|f_m\|\|\Delta^2\psi_m\| + c\big(\|\nabla\Delta\psi_m\| + \|\nabla\psi_m\|\big)\|\Delta^2\psi_m\| \\ &\quad \leqslant c\|f_m\|^2 + c\|\nabla\Delta\psi_m\|^2 + \frac{1}{2R}\|\Delta^2\psi_m\|^2. \end{aligned}$$

This shows the existence of two constant $m_0>1$ and c such that

$$\|\Delta^2P_m\psi_m\| = \|\Delta^2\psi_m\| \leqslant c\|f_m\| \quad \text{for } |m|\geqslant m_0.$$

The proof is complete. □

With the use of Lemma 3.1, we can now consider the solvability of periodic solutions to the linear evolutionary equation

$$\omega_{\mathrm{cr}}\partial_s\psi + \Delta^{-1}L_{\mathrm{cr}}\psi = \Delta^{-1}f \quad \text{with } s = t\omega_{\mathrm{cr}}. \tag{11}$$

It is convenient to use the notation

$$\begin{aligned} &\varphi_0 = e^{-is}\phi_0, \qquad \varphi_0^* = e^{is}\phi_0^*, \\ &\langle\phi,\psi\rangle = \frac{1}{2\pi}\int_0^{2\pi}(\phi,\psi)\,ds, \qquad \|\psi\|_{L_q} = \left(\frac{1}{2\pi}\int_0^{2\pi}\int_\Omega |\psi|^q\,dx\,dy\,ds\right)^{1/q}, \end{aligned}$$

the 2π-period complex function spaces

$$\mathbf{X} = \left\{\psi \in L_2\big(0, 2\pi; H^4\big);\ \psi(0) = \psi(2\pi),\ \|\psi\|_{\mathbf{X}} = \|\Delta\partial_s\psi\|_{L_2} + \left\|\Delta^2\psi\right\|_{L_2} < \infty\right\},$$
$$\mathbf{Y} = \left\{\psi \in L_2\big(0, 2\pi; H^2\big);\ \psi(2\pi) = \psi(0)\right\} \quad \text{with } \|\psi\|_{\mathbf{Y}} = \|\Delta\psi\|_{L_2},$$

and the projection operator

$$P\psi \equiv \psi - \langle\psi, \Delta\varphi_0^*\rangle\varphi_0 - \langle\psi, \Delta\bar{\varphi}_0^*\rangle\bar{\varphi}_0.$$

We thus have the Fredholm decompositions

$$\mathbf{X} = \operatorname{span}\{\varphi_0, \bar{\varphi}_0\} \oplus P\mathbf{X}, \qquad \mathbf{Y} = \operatorname{span}\{\varphi_0, \bar{\varphi}_0\} \oplus P\mathbf{Y}$$

with

$$P\mathbf{X} = \left\{\psi \in \mathbf{X};\ \langle\psi, \Delta\varphi_0^*\rangle = \langle\psi, \Delta\bar{\varphi}_0^*\rangle = 0\right\},$$

and

$$P\mathbf{Y} = \left\{\psi \in \mathbf{Y};\ \langle\psi, \Delta\varphi_0^*\rangle = \langle\psi, \Delta\bar{\varphi}_0^*\rangle = 0\right\}.$$

By applying P to (11), we have

$$\big(\omega_{\mathrm{cr}}\partial_s + \Delta^{-1}L_{\mathrm{cr}}\big)P\psi = P\Delta^{-1}f \quad \text{with } s = t\omega_{\mathrm{cr}}. \tag{12}$$

Solvability of this equation is given in the following lemma.

Lemma 3.2. *Let f be a 2π-periodic function in $L_2(0, 2\pi; L_2)$ or $\Delta^{-1}f \in \mathbf{Y}$. Then* (12) *admits a unique solution $\psi \in P\mathbf{X}$ such that $P\psi = \psi$ and*

$$\|\psi\|_{\mathbf{X}} = \left\|\big(\omega_{\mathrm{cr}}\partial_s + \Delta^{-1}L_{\mathrm{cr}}\big)^{-1}P\Delta^{-1}f\right\|_{\mathbf{X}} \leqslant c\|f\|_{L_2}.$$

Proof. We use the Fourier expansion

$$f(x, y, s) = \sum_{m=-\infty}^{\infty} f_m(x, y)e^{-ims}, \qquad \psi(x, y, s) = \sum_{m=-\infty}^{\infty} \psi_m(x, y)e^{-ims}$$

to transform (12) into the equivalent system of an infinite set of equations for any integer m:

$$\big(-im\omega_{\mathrm{cr}} + \Delta^{-1}L_{\mathrm{cr}}\big)P_m\psi_m = P_m\Delta^{-1}f_m$$

with P_m defined in Lemma 3.1. It follows from Lemma 3.1 that each of these equations is uniquely solvable and

$$\left\|\Delta^2 P_m\psi_m\right\| \leqslant c\|f_m\|,$$

where the constant c is independent of m. This gives

$$\left\|\Delta^2 P\psi\right\|_{L_2}^2 = \sum_{m=-\infty}^{\infty}\left\|\Delta^2 P_m\psi_m\right\|^2 \leqslant c\sum_{m=-\infty}^{\infty}\|f_m\|^2 = c\|f\|_{L_2}^2$$

and, by (12),

$$\|\partial_s\Delta P\psi\|_{L_2} \leqslant \|L_{\mathrm{cr}}P\psi\|_{L_2} + c\|f\|_{L_2} \leqslant c\left\|\Delta^2 P\psi\right\|_{L_2} + c\|f\|_{L_2} \leqslant c\|f\|_{L_2}.$$

The validity of $P\psi = \psi$ is obvious because of the projection property of P. The proof is complete. $\square$

Lemma 3.3. *The nonlinear operator $J(\cdot, \Delta\cdot)$ is a bounded mapping $\mathbf{X} \times \mathbf{X}$ into $L_2(0, 2\pi; L_2(\Omega))$. That is, for $\psi, \phi \in \mathbf{X}$, it is valid that*

$$\big\| J(\psi, \Delta\phi) \big\|_{L_2} \leqslant c\|\psi\|_{\mathbf{X}}\|\phi\|_{\mathbf{X}}.$$

Proof. For a domain D of an Euclidean space $\mathbf{R}^n$, a Banach space B, and an integer $k \geqslant 0$, we adopt the Bessel-potential spaces

$$H^k(D; B) = \left\{ \psi : D \mapsto B;\ \|\psi\|_{H^k} = \left(\sum_{m=1}^{k} \int_D \big\| \bar{\nabla}^m \psi \big\|_B^2 \, dx_1, \ldots, dx_n \right)^{1/2} < \infty \right\}$$

and

$$H^{k+\alpha}(D; B) = \big[H^k(D; B), H^{k+1}(D; B) \big]_\alpha, \quad 0 < \alpha < 1,$$

where $\bar{\nabla} \equiv (\partial_{x_1}, \ldots, \partial_{x_n})$ and $[\cdot, \cdot]_\alpha$ denote the complex interpolation functor (see, for example, Triebel [17,18]).

With the use of this notation, the definition of the space $\mathbf{X}$, and the complex interpolation space theory (see Triebel [17]), we have for $\alpha = 3/4$,

$$\begin{aligned} \mathbf{X} &\subset H^0\big(0, 2\pi; H^4(\Omega; R)\big) \cap H^1\big(0, 2\pi; H^2(\Omega; R)\big) \\ &\subset \big[H^0\big(0, 2\pi; H^4(\Omega; R)\big), H^1\big(0, 2\pi; H^2(\Omega; R)\big) \big]_\alpha \\ &= H^\alpha\big(0, 2\pi; H^{4(1-\alpha)+2\alpha}(\Omega; R)\big) = H^{3/4}\big(0, 2\pi; H^{2+1/2}(\Omega; R)\big), \end{aligned}$$

where $\subset$ denotes a continuous imbedding. Thus $\psi \in \mathbf{X}$ implies

$$\partial_x \psi, \partial_y \psi \in H^{3/4}\big(0, 2\pi; H^{1+1/2}(\Omega; R)\big) \subset L_\infty\big(0, 2\pi; L_\infty(\Omega; R)\big)$$

or

$$\|\nabla\psi\|_{L_\infty} \leqslant c\|\psi\|_{\mathbf{X}},$$

where we have used the Sobolev imbedding principle (see Triebel [17,18]). Therefore, for $\psi, \phi \in \mathbf{X}$, we have

$$\big\| J(\psi, \Delta\phi) \big\|_{L_2} \leqslant \|\nabla\psi\|_{L_\infty} \|\nabla\Delta\phi\|_{L_2} \leqslant c\|\psi\|_{\mathbf{X}}\|\phi\|_{\mathbf{X}}.$$

The proof is complete. □

4. Proof of Theorem 2.1

With the aid of the estimates derived in the previous section, we can now apply the implicit function theorem to prove Theorem 2.1 in the framework of Joseph and Sattinger [11]. To obtain a $2\pi/\omega$-periodic solution of (3) implies derivation of a 2π-periodic solution of the equation

$$\omega\partial_s\Delta\psi + L\psi + J(\psi, \Delta\psi) = 0, \quad s = \omega t. \tag{13}$$

Moreover, we seek the desired bifurcating periodic solution in the form

$$\psi = \varepsilon\tilde{\psi}, \qquad \tilde{\psi} = \varphi_0 + \bar{\varphi}_0 + \psi_1, \qquad \psi_1 = P\tilde{\psi}, \tag{14}$$

where ε is a small parameter such that

$$\omega = \omega(\varepsilon) \to \omega_{\mathrm{cr}}, \qquad r = r(\varepsilon) \to r_{\mathrm{cr}}, \quad \text{as } \varepsilon \to 0.$$

The substitution of this solution into (13) gives

$$\omega_{\mathrm{cr}} \partial_s \tilde{\psi} + \Delta^{-1} L_{\mathrm{cr}} \tilde{\psi} + (\omega - \omega_{\mathrm{cr}}) \partial_s \tilde{\psi} + (r - r_{\mathrm{cr}}) \tilde{\psi} + \varepsilon \Delta^{-1} J(\tilde{\psi}, \Delta \tilde{\psi}) = 0.$$

The application of the projection operator P on this equation and the use of (6) produce the equations

$$\big(\omega_{\mathrm{cr}} \partial_s + \Delta^{-1} L_{\mathrm{cr}}\big) P \tilde{\psi} + (\omega - \omega_{\mathrm{cr}}) \partial_s P \tilde{\psi} + (r - r_{\mathrm{cr}}) P \tilde{\psi} + \varepsilon P \Delta^{-1} J(\tilde{\psi}, \Delta \tilde{\psi}) = 0,$$

and

$$\big\langle (\omega - \omega_{\mathrm{cr}}) \partial_s \varphi_0 + (r - r_{\mathrm{cr}}) \varphi_0 + \varepsilon \Delta^{-1} J(\tilde{\psi}, \Delta \tilde{\psi}), \Delta \varphi_0^* \big\rangle = 0,$$

which provide the result

$$-i(\omega - \omega_{\mathrm{cr}}) + (r - r_{\mathrm{cr}}) + \varepsilon \big\langle J(\tilde{\psi}, \Delta \tilde{\psi}), \varphi_0^* \big\rangle = 0,$$

since

$$\big\langle \tilde{\psi}, \Delta \varphi_0^* \big\rangle = \big\langle \varphi_0, \Delta \varphi_0^* \big\rangle = \big(\phi_0, \Delta \phi_0^*\big) = 1.$$

It follows from Lemma 3.3 that

$$\big\| J(\tilde{\psi}, \Delta \tilde{\psi}) \big\|_{L_2} + \big| \big\langle J(\tilde{\psi}, \Delta \tilde{\psi}), \varphi_0^* \big\rangle \big| \leqslant \big\| J(\tilde{\psi}, \Delta \tilde{\psi}) \big\|_{L_2} \big(1 + \big\| \varphi_0^* \big\|_{L_2}\big) \leqslant c \|\tilde{\psi}\|_{\mathbf{X}}^2.$$

Taking Lemma 3.2 into account, we have

$$\left.\begin{aligned} 0 &= \psi_1 + \big(\omega_{\mathrm{cr}} \partial_s + \Delta^{-1} L_{\mathrm{cr}}\big)^{-1} \big[(\omega - \omega_{\mathrm{cr}}) \partial_s \psi_1 + (r - r_{\mathrm{cr}}) \psi_1\big] \\ &\quad + \varepsilon \big(\omega_{\mathrm{cr}} \partial_s + \Delta^{-1} L_{\mathrm{cr}}\big)^{-1} P \Delta^{-1} J(\varphi_0 + \bar{\varphi}_0 + \psi_1, \Delta \varphi_0 + \Delta \bar{\varphi}_0 + \Delta \psi_1), \\ 0 &= -(\omega - \omega_{\mathrm{cr}}) + \varepsilon \Im \big\langle J(\varphi_0 + \bar{\varphi}_0 + \psi_1, \Delta \varphi_0 + \Delta \bar{\varphi}_0 + \Delta \psi_1), \varphi_0^* \big\rangle, \\ 0 &= (r - r_{\mathrm{cr}}) + \varepsilon \Re \big\langle J(\varphi_0 + \bar{\varphi}_0 + \psi_1, \Delta \varphi_0 + \Delta \bar{\varphi}_0 + \Delta \psi_1), \varphi_0^* \big\rangle. \end{aligned}\right\} \tag{15}$$

Defining the right-hand side of this system by $F(\psi_1, \omega, r, \varepsilon)$, we see that

$$F : P\mathbf{X} \times \mathbf{R}^3 \mapsto P\mathbf{X} \times \mathbf{R}^2 \quad \text{with } F(0, \omega_{\mathrm{cr}}, r_{\mathrm{cr}}, 0) = 0$$

is a continuous operator. Furthermore, for $\phi_1 \in P\mathbf{X}$, this expression yields the following partial derivatives of F:

$$\begin{aligned} &\partial_{\psi_1} F(\psi_1, \omega, r, \varepsilon) \phi_1 \\ &\quad = \begin{pmatrix} \phi_1 + (\omega_{\mathrm{cr}} \partial_s + \Delta^{-1} L_{\mathrm{cr}})^{-1} [(\omega - \omega_{\mathrm{cr}}) \partial_s \phi_1 + (r - r_{\mathrm{cr}}) \phi_1] \\ \quad + \varepsilon (\omega_{\mathrm{cr}} \partial_s + \Delta^{-1} L_{\mathrm{cr}})^{-1} P \Delta^{-1} J(\phi_1, \Delta \varphi_0 + \Delta \bar{\varphi}_0 + \Delta \psi_1) \\ \quad + \varepsilon (\omega_{\mathrm{cr}} \partial_s + \Delta^{-1} L_{\mathrm{cr}})^{-1} P \Delta^{-1} J(\varphi_0 + \bar{\varphi}_0 + \psi_1, \Delta \phi_1) \\ \varepsilon \Im \langle J(\phi_1, \Delta \varphi_0 + \Delta \bar{\varphi}_0 + \Delta \psi_1) + J(\varphi_0 + \bar{\varphi}_0 + \psi_1, \Delta \phi_1), \varphi_0^* \rangle \\ \varepsilon \Re \langle J(\phi_1, \Delta \varphi_0 + \Delta \bar{\varphi}_0 + \Delta \psi_1) + J(\varphi_0 + \bar{\varphi}_0 + \psi_1, \Delta \phi_1), \varphi_0^* \rangle \end{pmatrix}, \end{aligned}$$

$$\partial_\omega F(\psi_1, \omega, r, \varepsilon) = \begin{pmatrix} (\omega_{\mathrm{cr}} \partial_s + \Delta^{-1} L_{\mathrm{cr}})^{-1} \partial_s \psi_1 \\ -1 \\ 0 \end{pmatrix},$$

$$\partial_r F(\psi_1, \omega, r, \varepsilon) = \begin{pmatrix} (\omega_{\mathrm{cr}} \partial_s + \Delta^{-1} L_{\mathrm{cr}})^{-1} \psi_1 \\ 0 \\ 1 \end{pmatrix}.$$

These results imply that the Fréchet derivative

$$\frac{\partial F(\psi_1, \omega, r, \varepsilon)}{(\psi_1, \omega, r)} : P\mathbf{X} \times \mathbf{R}^3 \mapsto L(P\mathbf{X}, P\mathbf{X}) \times \mathbf{R}^2$$

is continuous, and

$$\frac{\partial F(0, \omega_{\mathrm{cr}}, r_{\mathrm{cr}}, 0)}{\partial(\psi_1, \omega, r)} = \begin{pmatrix} I & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $L(P\mathbf{X}, P\mathbf{X})$ denotes the Banach space of bounded and linear operators mapping $P\mathbf{X}$ into itself and I represents the identity operator in the space $L(P\mathbf{X}, P\mathbf{X})$.

By the implicit function theorem (see, for example, [1]), there exists a unique solution $(\psi_1, \omega, r) \in P\mathbf{X} \times \mathbf{R}^2$ such that

$$\psi_1 = O(\varepsilon), \qquad \omega - \omega_{\mathrm{cr}} = O(\varepsilon), \qquad r - r_{\mathrm{cr}} = O(\varepsilon)$$

and $F(\psi_1, \omega, r, \varepsilon) = 0$ provided that $\varepsilon > 0$ is sufficiently small.

To show ψ_1 is a real function, we see that $\bar{\psi}_1 \in P\mathbf{X}$ satisfies the equation $F(\bar{\psi}_1, \omega - \omega_{\mathrm{cr}}, r - r_{\mathrm{cr}}, \varepsilon) = 0$ as well. By uniqueness, we obtain $\psi_1 = \bar{\psi}_1$. We have thus proved the uniqueness and existence of the real periodic function ψ_1, or the real periodic solution of (13):

$$\psi = \varepsilon(\varphi_0 + \bar{\varphi}_0) + \varepsilon\psi_1 \quad \text{with } \langle \psi_1, \varphi_0^* \rangle = \langle \psi_1, \bar{\varphi}_0^* \rangle = 0,$$

or the real periodic solution of (1), (2):

$$\psi = \psi_0 + \varepsilon(\varphi_0 + \bar{\varphi}_0) + \varepsilon\psi_1 \quad \text{with } \langle \psi_1, \varphi_0^* \rangle = \langle \psi_1, \bar{\varphi}_0^* \rangle = 0.$$

The proof of Theorem 2.1 is now complete.

5. Remarks

Theorem 2.1 shows the existence of bifurcating periodic solutions when the basic steady-state solution ψ_0 is independent of the Ekman number r. If ψ_0 is a function of r, an additional condition or a transversal crossing condition relating to varying Ekman number requires introduction to ensure the existence of periodic solutions. This result is given as follows.

Corollary 5.1. *In addition to the assumption imposed in Theorem* 2.1, *we assume*

$$\Re\left(\Delta\phi_0 + J\left(\phi_0, \Delta\frac{\partial\psi_0}{dr}\right) + J\left(\frac{\partial\psi_0}{dr}, \Delta\phi_0\right), \phi_0^* \right) \neq 0$$

or

$$1 + \Re\left(J\left(\phi_0, \Delta\frac{\partial\psi_0}{dr}\right) + J\left(\frac{\partial\psi_0}{dr}, \Delta\phi_0\right), \phi_0^* \right) \neq 0,$$

where the eigenfunction ϕ_0 and its conjugate eigenfunction ϕ_0^* are defined in Theorem 2.1. *Under this additional assumption the conclusion of Theorem* 2.1 *remains valid.*

To show this result, we write (13) with $\psi \in H^4$ in the following form:

$$0 = \omega\partial_s \Delta\psi + L_{\rm cr}\psi + (r - r_0)\Delta\psi + J\big(\psi, \Delta\psi_0(r) - \Delta\psi_0(r_{\rm cr})\big) \\ + J\big(\psi_0(r) - \psi_0(r_{\rm cr}), \Delta\psi\big) + J(\psi, \Delta\psi).$$

This is an equivalent formulation of (1), (2) or (13).

The substitution of solution (14) into this equation produces

$$0 = \omega_{\rm cr}\partial_s\tilde\psi + \Delta^{-1}L_{\rm cr}\tilde\psi + (\omega - \omega_{\rm cr})\partial_s\tilde\psi + (r - r_{\rm cr})\tilde\psi + \varepsilon\Delta^{-1}J(\tilde\psi, \Delta\tilde\psi) \\ + J\big(\tilde\psi, \Delta\psi_0(r) - \Delta\psi_0(r_{\rm cr})\big) + J\big(\psi_0(r) - \psi_0(r_{\rm cr}), \Delta\tilde\psi\big).$$

This equation has the extra term

$$J\big(\tilde\psi, \Delta\psi_0(r) - \Delta\psi_0(r_{\rm cr})\big) + J\big(\psi_0(r) - \psi_0(r_{\rm cr}), \Delta\tilde\psi\big) \tag{16}$$

due to the dependence of ψ_0 on r. Similar to the proof of Theorem 2.1, we apply the same projection operator P to this equation to obtain

$$\left.\begin{aligned} 0 = {} & \psi_1 + \big(\omega_{\rm cr}\partial_s + \Delta^{-1}L_{\rm cr}\big)^{-1}\big[(\omega - \omega_{\rm cr})\partial_s\psi_1 + (r - r_{\rm cr})\psi_1\big] \\ & + \varepsilon\big(\omega_{\rm cr}\partial_s + \Delta^{-1}L_{\rm cr}\big)^{-1}P\Delta^{-1}J(\varphi_0 + \bar\varphi_0 + \psi_1, \Delta\varphi_0 + \Delta\bar\varphi_0 + \Delta\psi_1) \\ & + \big(\omega_{\rm cr}\partial_s + \Delta^{-1}L_{\rm cr}\big)^{-1}P\Delta^{-1}J\big(\tilde\psi, \Delta\psi_0(r) - \Delta\psi_0(r_{\rm cr})\big) \\ & + \big(\omega_{\rm cr}\partial_s + \Delta^{-1}L_{\rm cr}\big)^{-1}P\Delta^{-1}J\big(\psi_0(r) - \psi_0(r_{\rm cr}), \Delta\tilde\psi\big), \\ 0 = {} & -\omega + \omega_{\rm cr} + \varepsilon\Im\big\langle J(\varphi_0 + \bar\varphi_0 + \psi_1, \Delta\varphi_0 + \Delta\bar\varphi_0 + \Delta\psi_1), \varphi^*\big\rangle \\ & + \Im\big\langle J\big(\tilde\psi, \Delta\psi_0(r) - \Delta\psi_0(r_{\rm cr})\big) + J\big(\psi_0(r) - \psi_0(r_{\rm cr}), \Delta\tilde\psi\big), \phi_0^*\big\rangle, \\ 0 = {} & r - r_{\rm cr} + \varepsilon\Re\big\langle J(\varphi_0 + \bar\varphi_0 + \psi_1, \Delta\varphi_0 + \Delta\bar\varphi_0 + \Delta\psi_1), \varphi^*\big\rangle \\ & + \Re\big\langle J\big(\tilde\psi, \Delta\psi_0(r) - \Delta\psi_0(r_{\rm cr})\big) + J\big(\psi_0(r) - \psi_0(r_{\rm cr}), \Delta\tilde\psi\big), \phi_0^*\big\rangle \end{aligned}\right\}$$

for $\tilde\psi = \varphi + \bar\varphi + \psi_1$. Let $\tilde F(\psi_1, \omega, r, \varepsilon)$ be the right-hand side of this equation. We see that $\tilde F(0, \omega_{\rm cr}, r_{\rm cr}, 0) = 0$, and $\tilde F$ has the same regularity property as the function F, the right-hand side of (15). In particular, the Fréchet derivative of $\tilde F$ is continuous and

$$\begin{aligned} & \frac{\partial\tilde F(0, \omega_{\rm cr}, r_{\rm cr}, 0)}{\partial(\psi_1, \omega, r)} \\ & = \begin{pmatrix} I & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 + \Re\big\langle J\big(\varphi_0 + \bar\varphi_0, \Delta\frac{\partial\psi_0}{dr}\big) + J\big(\frac{\partial\psi_0}{dr}, \Delta\varphi_0 + \Delta\bar\varphi_0\big), \varphi_0^*\big\rangle \end{pmatrix} \\ & = \begin{pmatrix} I & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 + \Re\big(J\big(\phi_0, \Delta\frac{\partial\psi_0}{dr}\big) + J\big(\frac{\partial\psi_0}{dr}, \Delta\phi_0\big), \phi_0^*\big) \end{pmatrix}, \end{aligned}$$

which is invertible due to the assumption admitted in the corollary. Thus the desired assertion is a consequence of the implicit function theorem.

The validity of the spectral simplicity condition and the spectral crossing condition relating to varying Reynolds number R were verified by Chen et al. [3] with respect to a mid-latitude atmospheric channel circulation model. The mid-axis of the channel is at 45° N, the channel extends approximately 360° in longitude, and the Ekman friction is not taken into consideration. Thus, as a consequence of the Hopf bifurcation theorem derived by Joseph and Sattinger [11], Chen et al. [3] derived the bifurcating periodic solutions branching off a critical value of Reynolds number.

For atmospheric wind-driven circulation models similar to the one discussed in [3] accounting for Ekman friction layers, we can also derive the validity of a spectral simplicity condition and the spectral crossing condition relating to varying Ekman number r and constant Reynolds number R. Therefore, the existence of bifurcating periodic solutions branching off a critical value of Ekman number with respect to the modified atmospheric wind-driven circulation models is a consequence of Theorem 2.1 and Corollary 5.1.

References

[1] M.S. Berger, Nonlinearity and Functional Analysis, Academic Press, San Diego, 1977.

[2] J. Charney, J. DeVore, Multiple flow equilibria in the atmosphere and blocking, J. Atmospheric Sci. 36 (1979) 1205–1216.

[3] Z.M. Chen, M. Ghil, E. Simonnet, S. Wang, Hopf bifurcation in quasi-geostrophic channel flow, SIAM J. Appl. Math. 64 (2003) 343–368.

[4] J. Duan, Time periodic quasigeotrophic motion under dissipation and forcing, Appl. Math. Comput. 102 (1999) 121–127.

[5] J. Duan, P.E. Kleoden, Dissipative quasigeostrophic motion under temporally periodic forcing, J. Math. Anal. Appl. 236 (1999) 74–85.

[6] M. Ghil, S. Childress, Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory, and Climate Dynamics, Springer-Verlag, Berlin, 1987.

[7] E. Hopf, A mathematical example displaying features of turbulence, Comm. Pure Appl. Math. 1 (1948) 303–322.

[8] E. Hopf, Repeated branching through loss of stability, an example, in: Proc. Conf. on Diff. Equations, Univ. of Maryland, 1955.

[9] S. Jiang, F.-F. Jin, M. Ghil, Multiple equilibria, periodic, and aperiodic solutions in a wind-driven, double-gyre, shallow-water model, J. Phys. Oceanogr. 25 (1995) 764–786.

[10] F.-F. Jin, M. Ghil, Intraseasonal oscillations in the extratropics: Hopf bifurcation and topographic instabilities, J. Atmospheric Sci. 47 (1990) 3007–3022.

[11] D.D. Joseph, D. Sattinger, Bifurcating time periodic solutions and their stability, Arch. Rational Mech. Anal. 45 (1972) 75–109.

[12] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon & Breach, New York, 1968.

[13] J.L. Lions, R. Temam, S. Wang, New formulations of the primitive equations of atmosphere and applications, Nonlinearity 5 (1992) 237–288.

[14] J.L. Lions, R. Temam, S. Wang, On the equations of large-scale ocean, Nonlinearity 5 (1992) 1007–1053.

[15] J. Pedlosky, Resonant topographic waves in barotropic and baroclinic flows, J. Atmospheric Sci. 38 (1981) 2626–2641.

[16] J. Pedlosky, Geophysical Fluid Dynamics, second ed., Springer-Verlag, Berlin, 1987.

[17] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.

[18] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, 1983.

[19] G. Veronis, An analysis of wind-driven ocean circulation with a limited number of Fourier components, J. Atmospheric Sci. 20 (1963) 577–593.

[20] G. Veronis, Wind-driven ocean circulation. Part II: Numerical solution of the nonlinear problem, Deep-Sea Res. 13 (1966) 30–55.
[21] K. Yosida, Functional Analysis, Springer-Verlag, Berlin, 1965.
[22] V.I. Yudovich, The onset of auto-oscillations in a fluid, J. Math. Mech. 35 (1971) 638–655.